Modeling the impact of government guarantees on toll charge, road quality and capacity for Build-Operate-Transfer (BOT) road projects

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A B S T R A C T

Government guarantees are frequently used to attract private investors' participation into Build-Operate-Transfer (BOT) road projects. In this paper, we investigate the impact of government guarantees on toll charge, road quality and road capacity by taking perspective of the private investor. The main results are: (1) Minimum traffic guarantee increases toll charge while decreasing road quality. Under a low guarantee level, minimum traffic guarantee has no impact on road capacity. However, it improves road capacity when a high guarantee level is performed. (2) Under minimum revenue guarantee, if the guarantee level is sufficiently high, the optimal toll charge will be sufficiently large, but road quality and road capacity will approach zero. (3) Price compensation guarantee decreases toll charge and increases both road quality and road capacity. This paper further investigates the impact of government guarantees when the contract is auctioned. We find that auction reduces the impact of government guarantees on toll charge while failing to affect the impact of government guarantees on road quality and capacity. Some policy implications are also derived from our model results.

1. Introduction

In the last two decades, public roads have been increasingly provided by private investors via the Build-Operate-Transfer (BOT) approach (Tan, 2012; Thomas et al., 2006). In this approach, a private investor builds and operates a road within a defined period and transfers the ownership at no cost to the government at the end of the contract (Nombela and de Rus, 2004; Wu et al., 2011; Xiao et al., 2007). For the BOT road project, the private investor recoups investment mainly through toll revenue during the operation stage. However, in many cases, the private investor cannot obtain a positive profit solely by toll revenue due to huge investment required by the project. Therefore, to encourage the private investor's participation, the government offers various government guarantees in practice (Irwin, 2007).

There is a growing literature on government guarantees in BOT road projects. One stream of research focuses on how government guarantees can reduce project risks. Wibowo (2004) studied the financial impact of government guarantees on project cash flows and concluded that government guarantees were more effective than direct subsidies in reducing project risks. Vassallo and Solino (2006) specifically investigated the mechanism of minimum revenue guarantee in mitigating traffic risk. They argued that this guarantee type encouraged private investors' participation while incurring low fiscal impact on

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governments. Wibowo and Kochendoerfer (2011, 2012) further developed a framework to quantify the fiscal impact of government guarantees. Another stream of research investigates the optimal guarantee level. For instance, by taking into account shadow cost of public funds and private investors’ risk aversion, Engel et al. (2013) gave the optimal guarantee level under minimum revenue guarantee for intermediate-demand road projects. Finally, there is a number of studies undertaken recently to examine the option value of government guarantees by employing real option models (Ashuri et al., 2012; Brandao and Saraiva, 2008; Galera and Solino, 2010).

Even though many facets of government guarantees have been studied, to our best knowledge, the impact of government guarantees on the project is largely overlooked by previous literature. In this paper, we aim to fill this theoretical gap by investigating whether and, if so, how the project can be affected by different government guarantees. Specifically, we will investigate the impact of government guarantees – minimum traffic guarantee, minimum revenue guarantee, and price compensation guarantee – on three project characteristics: toll charge, road quality, and road capacity. These three characteristics are important variables in the private investor’s profit function. To be specific, higher toll charge increases toll revenue per user; however, it also reduces traffic demand. Higher road quality can bring higher demand and decrease operating and maintenance (O&M) costs, but can incur higher construction cost. Higher road capacity satisfies traffic demand with a greater probability when the demand is uncertain, but can also result in higher capacity investment.

To understand the impact of government guarantees on the above three variables, we conduct our analysis in three steps. In the first step, we analyze the benchmark case that government guarantees are absent and there is only one private investor. In the second step, we incorporate government guarantees into our analysis. In the last step, we investigate whether and, if so, how the impact of government guarantees on the project would change when the contract is auctioned. In this process, the impact of government guarantees is studied separately on toll charge, road quality, and road capacity, that is, we look at only one variable at a time by holding the other two variables fixed.

This paper proceeds as follows. Section 2 presents model assumptions. Section 3 analyzes the benchmark case in our model. Section 4 analyzes the impact of three government guarantee types – minimum traffic guarantee, minimum revenue guarantee, and price compensation guarantee – on toll charge, road quality, and road capacity. Section 5 further incorporates auction into our model. Section 6 gives a simple numerical example to demonstrate our model results. Section 7 summarizes our results with some policy suggestions. We relegate all proofs to the Appendix A.

2. Model assumptions

Consider a new BOT road project that has no other competitive roads nearby. Following Wu et al. (2011) and Xiao et al. (2007), we assume that the private investor behaves as a profit maximizer. She determines toll charge, road quality, and road capacity to maximize her expected profit. Let us denote toll charge, road quality, and road capacity by \( p, q, y \), respectively. Based on Chen and Subprasom’s (2007) suggestion, we assume that the private investor faces stochastic traffic demand, which is a minor extension of previous studies that were based on deterministic traffic demand (Nombela and de Rus, 2004; Subprasom and Chen, 2007; Yang and Meng, 2000, 2002).

Let \( d(p, q) \) denote the stochastic traffic demand that depends on toll charge \( (p) \) and road quality \( (q) \) (Qiu and Wang, 2011). \( d(p, q) \) cannot exceed \( D \), where \( D \) is sufficiently large. Suppose that the cumulative density distribution of \( d(p, q) \) is \( F(x, p, q) \), where \( F(x, p, q) \) represents probability that traffic demand is lower than \( x \) and \( F(0, p, q) = 0 \). \( F(D, p, q) = 1 \). Traffic demand is strictly decreasing in \( p \) and increasing in \( q \). Thus, it is reasonable to assume that \( F_x(x, p, q) < 0 \) and \( F_{xx}(x, p, q) > 0 \), where \( F_x(x, p, q) = 1 - F(x, p, q) \) and \( F_y(x, p, q) \) mean partial derivatives of \( F(x, p, q) \) with respect to \( p \) and \( q \), respectively.

When road quality is \( q \) and road capacity is \( y \), construction cost can be written as \( C + \int_0^q \int_0^y \omega(s, x)dx \) (Yang et al., 2002; Qiu and Wang, 2011), where \( C \) and \( \omega(q, y) \) represent the fixed and marginal construction cost, respectively. Operating and maintenance (O&M) costs per user depend on road quality. If we denote O&M costs per user by \( c_0(q) \), then according to Hoppe et al. (2013), we have \( c_0'(q) < 0 \) and \( \| f_{(q)}(mnyd\|p,q)\| < 0 \), that is, higher road quality results in lower O&M costs per user and lower total O&M costs. For tractability, we assume that the private investor’s expected profit is strictly concave with respect to \( p, q \), and \( y \) to guarantee interior solutions to the private investor’s optimization problem. Finally, suppose that the private investor’s reservation utility is \( \Pi \).

3. Benchmark case: single private investor without government guarantees

In this case, the private investor’s expected profit is defined as toll revenue minus construction and O&M costs. The discount factor is denoted by \( \delta \). We assume that construction and operation periods will last \( m \) and \( n \) years, respectively. Then the private investor’s expected profit is

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1. According to Walker and Smith (1995), 50% prolongation of highway life requires additional 2–5% costs on the materials for the subgrade and pavement.

2. According to Fields et al. (2009), “insufficient capacity accounts for about one-third of congestion problems” in very large urban areas.

3. The private investor may sometimes seek to expand geographically or just maintain a steady workflow in times of economic recession. But in general, profit maximization is often her main goal. We thank one anonymous reviewer for this point.
\[\Pi_0(p,q,y) = (\delta^{m+1} + \delta^{m+2} + \cdots + \delta^{m+n}) p \min(y, d(p,q)) - \left( \delta \int_{y_1}^{y_2} \int_{q_1}^{q_2} \omega(s,x) \, ds \, dx \right.
\]
\[+ \delta^2 \int_{y_1}^{y_2} \int_{q_1}^{q_2} \omega(s,x) \, ds \, dx + \cdots + \delta^m \int_{y_{m-1}}^{y_m} \int_{q_{m-1}}^{q_m} \omega(s,x) \, ds \, dx \left. \right) - (\delta^{m+1} + \delta^{m+2} + \cdots + \delta^{m+n}) c_o(q) \min(y, d(p,q)).\]

where \(y_i\) and \(q_i\) (\(i = 1, \ldots, m\)) represent road capacity and road quality, respectively, at the end of each year in the construction period. The first item is the private investor’s toll revenue during the operation stage, and the last two items are construction and O&M costs, respectively. We have dropped the fixed construction cost \(C\) in the above program for it does not affect our analysis. However, it must be noticed that to encourage the private investor’s participation, the fixed construction cost also need to be recovered.

We assume that \(0 < \max_{p,q,y} \Pi_0(p,q,y) < \Pi\). This means that even though toll revenue can recover construction and O&M costs, it cannot make the private investor achieve her reservation utility. As a result, government guarantees have to be introduced to attract the private investor’s participation.

To make the program elegant, we replace \((\delta^{m+1} + \delta^{m+2} + \cdots + \delta^{m+n})\) with \(z\) and let \(\overline{\omega}(q,y) = \delta \int_{0}^{\infty} \omega(s,x) \, ds \, dx + \cdots + \delta^m \int_{y_{m-1}}^{y_m} \omega(s,x) \, ds \, dx\), then \(\Pi_0(p,q,y)\) can be rewritten as

\[\Pi_0(p,q,y) = z(p - c_o(q)) \min(y, d(p,q)) - \overline{\omega}(q,y) = z(p - c_o(q)) \left[ \int_{0}^{\infty} x F(x,p,q) + y F(y,p,q) \right] - \overline{\omega}(q,y),\]

where the last equality is derived by integration by parts.

Taking derivative of \(\Pi_0(p,q,y)\) with respect to \(p\), we have

\[k_0(p) = \frac{\partial \Pi_0(p,q,y)}{\partial p} = z \int_{0}^{\infty} F(x,p,q) \, dx + z(p - c_o(q)) \int_{0}^{\infty} F_p(x,p,q) \, dx.\]

Thus, the optimal toll charge \(p_0\) in the benchmark case satisfies \(k_0(p_0) = 0\).

Taking derivative of \(\Pi_0(p,q,y)\) with respect to \(q\), we have

\[h_0(q) = \frac{\partial \Pi_0(p,q,y)}{\partial q} = -z c_o'(q) \int_{0}^{\infty} F(x,p,q) \, dx + z(p - c_o(q)) \int_{0}^{\infty} F_q(x,p,q) \, dx - \delta^m \int_{y_{m-1}}^{y_m} \omega(q,x) \, dx.\]

If we denote the optimal road quality in the benchmark case by \(q_0\), then we have \(h_0(q_0) = 0\).

For any given \(p\) and \(q\), we now investigate the optimal road capacity. Taking derivative of \(\Pi_0(p,q,y)\) with respect to \(y\) gives

\[g_0(y) = \frac{\partial \Pi_0(p,q,y)}{\partial y} = z(p - c_o(q)) F(y,p,q) - \delta^m \int_{y_{m-1}}^{y_m} \omega(s,y) \, ds.\]

Similarly, the optimal road capacity \(y_0\) satisfies \(g_0(y_0) = 0\).

Therefore, the private investor’s optimal toll charge, road quality, and road capacity in the benchmark case are \(p_0\), \(q_0\), and \(y_0\), respectively.

4. The private investor’s decisions under government guarantees

In this section, we will investigate the private investor’s optimal decisions on toll charge, road quality, and road capacity under three common guarantee types in practice: minimum traffic guarantee, minimum revenue guarantee, and price compensation guarantee (Wibowo, 2004).

4.1. Minimum traffic guarantee

Under minimum traffic guarantee, the government guarantees the minimum traffic level. The government pays nothing to the private investor if the actual traffic is equal or higher than the guarantee level; otherwise the government compensates the private investor for unrealized traffic (Wibowo, 2004). Let \(\nu\) denote the guaranteed traffic level. Suppose that the private investor’s expected profit in this case is \(\Pi'_1(p,q,y)\). Moreover, we should have \(\Pi'_1(p,q,y) > \Pi\) for some \(p,q\), and \(y\). Otherwise minimum traffic guarantee cannot play a role in attracting the private investor. Moreover, road capacity should not be lower than the guarantee level, i.e., \(y \geq \nu\). When the realized traffic is lower than \(\nu\), the government “buybacks” the

\[\text{Note that in this model, we treat } y \text{ as a continuous variable; however, this can be easily extended to the discrete version. Because we assume that } \Pi_0(p,q,y) \text{ is strictly concave with respect to } y, \text{ the private investor can always compare two integers around } y_0 \text{ to choose the one that makes her obtain a higher profit. But for convenience, we still employ } y_0.\]
unrealized traffic with price $b$ that satisfies $0 < b \leq p - c_v(q)$, that is, the price should be lower than the private investor’s profit per user. Otherwise, to increase her profit, the private investor will depress traffic demand to get more government compensation. The private investor’s expected profit under minimum traffic guarantee can be written as

$$
\Pi_1^v(p, q, y) = 2p \min(y, d(p, q)) - \overline{c}(q, y) - \alpha c_v(q) \min(y, d(p, q)) + \alpha b \mathbb{E}(v - d(p, q))
$$

$$
= \alpha(p - c_v(q)) \int_0^y f(x, p, q)dx - \overline{c}(q, y) + \alpha b \int_0^y (v - x)f(x, p, q)
$$

$$
= \alpha(p - c_v(q)) \int_0^y f(x, p, q)dx - \overline{c}(q, y) + \alpha b \int_0^y f(x, p, q)dx,
$$

where $(v - d(p, q)) = \begin{cases} v - d(p, q) & d(p, q) > v \\ 0 & d(p, q) \leq v \end{cases}$ and the last equality is derived by integration by parts.

Taking derivative of $\Pi_1^v(p, q, y)$ with respect to $p$, we can obtain that

$$
k_1^v(p) = \frac{\partial \Pi_1^v(p, q, y)}{\partial p} = \alpha \int_0^y f(x, p, q)dx + \alpha(p - c_v(q)) \int_0^y F_v(x, p, q)dx + \alpha b \int_0^y F_p(x, p, q)dx.
$$

If the optimal toll charge under minimum traffic guarantee is denoted by $p_1^v(v)$, then we have $k_1^v(p_1^v(v)) = 0$.

Taking derivative of $\Pi_1^v(p, q, y)$ with respect to $q$ gives

$$
h_1^v(q) = -\alpha c_v(q) \int_0^y f(x, p, q)dx + \alpha(p - c_v(q)) \int_0^y F_v(x, p, q)dx - \delta \int_{y_{m-1}}^y \omega(q, x)dx + \alpha b \int_0^y F_q(x, p, q)dx.
$$

If the optimal road quality under minimum traffic guarantee is denoted by $q_1^v(v)$, then we have $h_1^v(q_1^v(v)) = 0$.

We now allow the private investor to determine the road capacity. Naturally, the optimal road capacity should not be lower than the guarantee level. Taking derivative of $\Pi_1^v(p, q, y)$ with respect to $y$, we have

$$
g_1^v(y) = \alpha(p - c_v(q))f(y, p, q) - \delta \int_{y_{m-1}}^y \omega(s, y)ds.
$$

If $y_1^v(v)$ satisfies $g_1^v(y_1^v(v)) = 0$, then based on the assumption of concave profit function, we can derive that the optimal road capacity under minimum traffic guarantee is $y_1^v(v) = \max(y_1^v(v), v)$.

**Proposition 1.** $p_1^v(v)$ is increasing in $v, q_1^v(v)$ is decreasing in $v, y_1^v(v)$ is independent of $v$.

**Proposition 1** shows that higher guarantee level results in higher toll charge and lower road quality under minimum traffic guarantee. The logic behind this argument is as follows. If the government raises the guarantee level, the realized demand will be more likely to fall below this guarantee level, which makes the private investor less concerned about traffic demand. Therefore, the private investor will be also less concerned about the impact of toll charge and road quality on traffic demand. As a result, she will be inclined to increase toll charge to increase her profit per user and to lower road quality to decrease her construction cost. Road capacity $y$ can be interpreted as the maximum demand. Note that for any given $p$ and $q$, the maximum traffic demand is not affected by $v$ due to $y \geq v$. Thus, $y_1^v(v)$ is independent of $v$.

**Proposition 2.** Under minimum traffic guarantee, we have $p_1^v(v) > p_0^v, q_1^v(v) < q_0^v$ for any guarantee level $v$. Moreover, when $v \leq y_0^v$, we have $y_1^v(v) = y_0^v$; otherwise, $y_1^v(v) > y_0^v$.

**Proposition 2** shows that, compared with the benchmark case, minimum traffic guarantee provides incentives for the private investor to increase toll charge and decrease road quality. Moreover, if the guarantee level is low ($v \leq y_0^v$), the private investor’s optimal road capacity will not change. However, if the guarantee level is high ($v > y_0^v$), the private investor’s optimal road capacity will become $v$, which is higher than that in the benchmark case.

### 4.2 Minimum revenue guarantee

Under minimum revenue guarantee, the government pays nothing to the private investor if the realized toll revenue is equal or higher than the guarantee level; otherwise, the government compensates the private investor for the unrealized revenue (Huang and Chou, 2006; Wisbowski, 2004).

Let us denote the revenue guarantee level by $R$. Note that the private investor’s toll revenue is $2p \min(y, d(p, q))$ when there are no government guarantees. Thus, if $2p \min(y, d(p, q)) > R$, the private investor’s expected profit in this case will still be $\Pi_0^v(p, q, y)$, which makes minimum revenue guarantee fail to attract the private investor; otherwise if $2p \min(y, d(p, q)) \leq R$, the private investor’s expected profit $\Pi_2^v(p, q, y)$ will be

$$
\Pi_2^v(p, q, y) = R - \overline{c}(q, y) - \alpha c_v(q) \min(y, d(p, q)).
$$

The following proposition characterizes the private investor’s optimal decisions in this case.

**Proposition 3.**
Proposition 3. If \( \max_q \{\alpha \min(y, d(p, q))\} \leq R \) for any given \( q \) and \( y \), then the optimal toll charge will be sufficiently large; the optimal road quality and road capacity will approach zero.

Proposition 3 shows that minimum revenue guarantee is effective in attracting the private investor only when the guarantee level is sufficiently high. With a high revenue guarantee level, the government will undertake all revenue risks. As a result, the private investor will concentrate on minimizing her construction and O&M costs by reducing road quality and road capacity and increasing toll charge (thus decreasing traffic demand). Therefore, minimum revenue guarantee encourages the private investor to offer high toll charge and low road quality and capacity. Moreover, Proposition 3 also shows that when minimum revenue guarantee is effective, the private investor’s optimal toll charge, road quality, and road capacity stay the same under different guarantee levels.

4.3. Price compensation guarantee

Under price compensation guarantee, the government pays the private investor a fixed price \( u \) for each unit of realized demand (Ye and Liu, 2008). This guarantee type is employed in the BOT-based Line 4 of Beijing Metro, in which toll revenue is insufficient to make the private investor obtain her reservation utility. Similarly, we assume that the private investor’s expected profit under price compensation guarantee \( \Pi^3(p, q, y) \) satisfies \( \Pi^3(p, q, y) > \Pi \) for some \( p, q, \) and \( y \). \( \Pi^3(p, q, y) \) can be written as

\[
\Pi^3(p, q, y) = \alpha(p + u) \min(y, d(p, q)) - \sigma(q, y) - \alpha c_o(q) \min(y, d(p, q)) = \alpha(p + u - c_o(q)) \int_0^y F(x, p, q) dx - \sigma(q, y).
\]

Taking derivative of \( \Pi^3(p, q, y) \) with respect to \( p \), we can get

\[
k_3^3(p) = \frac{\partial \Pi^3(p, q, y)}{\partial p} = \alpha \int_0^y F(x, p, q) dx + \alpha(p + u - c_o(q)) \int_0^y F_p(x, p, q) dx.
\]

Then the optimal toll charge under price compensation guarantee \( p_3^3(u) \) satisfies \( k_3^3(p_3^3(u)) = 0 \).

Taking derivative of \( \Pi^3(p, q, y) \) with respect to \( q \) gives

\[
h_3^3(q) = \frac{\partial \Pi^3(p, q, y)}{\partial q} = -\alpha c_o'(q) \int_0^y F(x, p, q) dx + \alpha(p + u - c_o(q)) \int_0^y F_q(x, p, q) dx - \delta_m \int_{y_m}^y \omega(q, x) dx.
\]

Then the optimal road quality under price compensation guarantee \( q_3^3(u) \) satisfies \( h_3^3(q_3^3(u)) = 0 \).

Taking derivative of \( \Pi^3(p, q, y) \) with respect to \( y \), we have

\[
g_3^3(y) = \frac{\partial \Pi^3(p, q, y)}{\partial y} = \alpha(p + u - c_o(q)) F(y, p, q) - \delta_m \int_{y_m}^y \omega(s, y) ds.
\]

Then the optimal road capacity under price compensation guarantee \( y_3^3(u) \) satisfies \( g_3^3(y_3^3(u)) = 0 \).

Proposition 4. \( p_3^3(u) \) is decreasing in \( u \); \( q_3^3(u) \) and \( y_3^3(u) \) are increasing in \( u \).

Proposition 4 indicates that higher guarantee level results in lower toll charge, higher road quality and road capacity. Indeed, note that when the guarantee level \( u \) is high, the private investor can obtain more benefits by increasing traffic demand. Therefore, to increase traffic demand, the private investor will be encouraged to decrease toll charge and increase both road quality and road capacity.

Proposition 5. Under price compensation guarantee, we have \( p_3^3(u) < p_0^3 \), \( q_3^3(u) > q_0^3 \), and \( y_3^3(u) > y_0^3 \) for any guarantee level \( u \).

Proposition 5 shows that, compared with the benchmark case, price compensation guarantee decreases toll charge, and increases both road quality and road capacity. Therefore, price compensation guarantee can serve as an incentive mechanism for the private investor to invest in road quality and road capacity.

In practice, many guarantee types are essentially price compensation guarantee. For example, to increase profitability of a BOT road project, the government may grant the private investor rights to develop land alongside the road (Ye and Liu, 2008). Daxing Line of Beijing Subway adopts such a guarantee type. In this project, the private investor recovers her investment by toll revenue and revenue from land development, the latter accounting for approximately 90% of investment (Wang, 2010, p.131). If we assume that the land value is positively related to the traffic demand \( (V = \beta \min(y, d(p, q))) \), then this guarantee type can be immediately transformed into price compensation guarantee by substituting \( u \) with \( \beta \).

5. The private investor’s decisions when the contract is auctioned

In this section, we will investigate whether and, if so, how the private investor’s decisions on toll charge, road quality, and road capacity will change when the BOT contract is auctioned. Suppose that the government awards the project to the private investor with the lowest toll charge. Actually, this is a common practice for BOT roads (Guasch, 2004, p.98). In this
paper, we rule out the possibility that investors make aggressive bids and subsequently initiate opportunistic renegotiation by informational advantage.

Let us denote the probability density function and cumulative density function of toll charge by \( t(p) \) and \( T(p) \), respectively. Then the hazard rate can be defined as \( \lambda(p) = \frac{t(p)}{T(p)} = \frac{\partial T(p)}{\partial p} \), where \( \overline{T}(p) = 1 - T(p) \). As a common assumption of hazard rate, suppose that \( \lambda(p) \) is strictly increasing in \( p \) (Zheng, 2002). For tractability, we still assume that the private investor's expected profit function is strictly concave with respect to \( p, q, \) and \( y \).

5.1. The contract is auctioned without government guarantees

When the contract is auctioned, the probability that a specific private investor wins out can be represented as \( \int_0^\infty t(r)dr = T(p) \). Then this private investor’s expected profit will be

\[
\Pi_0(p, q, y) = \Pi_0(p, q, y) \int_0^\infty t(r)dr = \Pi_0(p, q, y) T(p).
\]

Recall that \( \Pi_0(p, q, y) < \Pi \) for any \( p, q, \) and \( y \). Thus, we also have \( \Pi_0(p, q, y) < \Pi \). This means that government guarantees are still necessary to attract the private investor’s participation under auction. Taking derivative of \( \Pi_0(p, q, y) \) with respect to \( p \) gives

\[
k_0(p) = \frac{\partial \Pi_0(p, q, y)}{\partial p} T(p) - \Pi_0(p, q, y) t(p) = k_0(p) T(p) - \Pi_0(p, q, y) t(p).
\]

Recall that \( k_0(p_0) = 0 \). Thus, we can obtain that \( k_0(p_0) = -\Pi_0(p_0, q, y) t(p_0) < 0 \). If we denote the optimal toll charge in this case by \( p_0 \), then \( k_0(p_0) = 0 \). Based on our assumption that \( \Pi_0(p, q, y) \) is strictly concave with respect to \( p \), we can get \( p_0 < p_0 \), which means that the private investor's optimal toll charge is decreased under auction compared with the benchmark case.

Note that the probability to win out is independent of \( q \) and \( y \). Thus, we can immediately apply the above results of \( q \) and \( y \) to this section. Therefore, in our following analysis, we will not investigate the private investor’s decisions on road quality and capacity.

**Proposition 6.** When the contract is auctioned, the private investor’s optimal toll charge is decreased compared with the benchmark case. However, auction does not affect the private investor’s decisions on road quality and road capacity.

5.2. Minimum traffic guarantee under auction

When the contract is auctioned, the private investor’s expected profit under minimum traffic guarantee will be \( \Pi^*_1(p, q, y) = \Pi^*_1(p, q, y) T(p) \). We assume that with the guarantee level of \( v \), \( \Pi^*_1(p, q, y) > \Pi \) for some \( p, q, \) and \( y \). Taking derivative of \( \Pi^*_1(p, q, y) \) with respect to \( p \) gives

\[
k^*_1(p) = \frac{\partial \Pi^*_1(p, q, y)}{\partial p} T(p) - \Pi^*_1(p, q, y) t(p) = k^*_1(p) T(p) - \Pi^*_1(p, q, y) t(p).
\]

Similarly, the optimal toll charge \( \tilde{p}_1^*(v) \) satisfies \( \tilde{k}^*_1(\tilde{p}_1^*(v)) = 0 \) and \( \tilde{p}_1^*(v) < p_1^*(v) \). Moreover, we have the following proposition.

**Proposition 7.** If \( F(v, p, q) T(p) \) is strictly increasing in \( p \), then both \( \tilde{p}_1^*(v) \) and \( p_1^*(v) - \tilde{p}_1^*(v) \) are also strictly increasing in \( v \).

Recall that \( T(p) \) is the probability to win out. Thus, \( F(v, p, q) T(p) \) can be interpreted as a private investor’s “expected” probability that traffic demand falls below \( v \) when her toll charge is \( p \), which justifies our condition that \( F(v, p, q) T(p) \) strictly increases with \( p \). Under such a condition, we can obtain that the optimal toll charge under auction increases with the guarantee level.

Suppose that \( v_1 \leq v_2 \), then Proposition 7 gives \( \tilde{p}_1^*(v_1) - \tilde{p}_1^*(v_2) \leq \tilde{p}_1^*(v_2) - \tilde{p}_1^*(v_2) \) or \( \tilde{p}_1^*(v_2) - \tilde{p}_1^*(v_1) \geq \tilde{p}_1^*(v_2) - \tilde{p}_1^*(v_1) \). Thus, the change of optimal toll charge with guarantee levels becomes smaller when the contract is auctioned. This means that the impact of minimum traffic guarantee on the optimal toll charge is reduced in the presence of auction. Moreover, Proposition 7 shows that when \( v \) is higher, the private investor’s optimal toll charge will be more reduced under auction. Therefore, the effect of auction on reducing toll charge increases with the guarantee level.

5.3. Minimum revenue guarantee under auction

If \( xp \min(y, d(p, q)) > R \), then the private investor’s expected profit will be \( \Pi_0(p, q, y) \), which is insufficient to make her achieve her reservation utility. As a result, minimum revenue guarantee fails to attract the private investor under auction.
Otherwise if \( xp \min(y, d(p, q)) \leq R \), the private investor’s expected profit will be \( \Pi^2_{p}(p, q, y) = \Pi^2_{p}(p, q, y)T(p) \). Taking derivative of \( \Pi^2_{p}(p, q, y) \) with respect to \( p \) gives

\[
\frac{\partial \Pi^2_{p}(p, q, y)}{\partial p} T(p) - \Pi^2_{p}(p, q, y)t(p).
\]

Suppose that \( \frac{\partial \Pi^2_{p}(p, R)}{\partial p} = 0 \), then we have the following proposition.

**Proposition 8.** When the contract is auctioned, if \( \max_y(x, d(p, q)) \leq R \) for any given \( q \) and \( y \), then the optimal toll charge will be \( p^*_2(R) \) under minimum revenue guarantee. Moreover, \( p^*_2(R) \) is decreasing in \( R \).

In comparison with Proposition 3, we can see that auction reduces the private investor’s optimal toll charge under minimum revenue guarantee. Moreover, the optimal toll charge in this case will depend on the guarantee level. Proposition 8 further shows that higher guarantee level results in lower toll charge. This is because when the private investor’s toll revenue is better guaranteed, she will decrease the toll charge to increase her probability to win the auction.

5.4. Price compensation guarantee under auction

The private investor’s expected profit in this case will be \( \Pi^3_{p}(p, q, y) = \Pi^3_{p}(p, q, y)T(p) \). We assume that with guarantee level \( u \), \( \Pi^3_{p}(p, q, y) \) satisfies \( \Pi^3_{p}(p, q, y) > \Pi \) for some \( p \), \( q \), and \( y \). Taking derivative of \( \Pi^3_{p}(p, q, y) \) with respect to \( p \) gives

\[
\frac{\partial \Pi^3_{p}(p, q, y)}{\partial p} T(p) - \Pi^3_{p}(p, q, y)t(p) = k^3_{p}(p)T(p) - \Pi^3_{p}(p, q, y)t(p).
\]

Similarly, the optimal toll charge satisfies \( p^*_3(u) < p^*_2(u) \) and \( k^3_{p}(p^*_3(u)) = 0 \). Moreover, we have the following proposition.

**Proposition 9.** Both \( p^*_1(u) \) and \( p^*_2(u) \) are decreasing in \( u \).

Still, the optimal toll charge under price compensation guarantee decreases with the guarantee level in the presence of auction. Suppose that \( u_1 \leq u_2 \), then Proposition 9 gives \( p^*_1(u_1) - p^*_1(u_2) \geq p^*_2(u_1) - p^*_2(u_2) \) or \( p^*_3(u_1) - p^*_3(u_2) \geq p^*_3(u_1) - p^*_3(u_2) \).

Thus, the change of optimal toll charge with guarantee levels becomes smaller under auction. This means that auction reduces the impact of price compensation guarantee on toll charge. Moreover, note that higher \( u \) results in smaller difference between \( p^*_3(u) \) and \( p^*_3(u) \). Thus, the effect of auction on reducing toll charge decreases with the guarantee level.

6. Numerical analysis

To test our model results, we present a simple numerical example in this section. We only analyze the impact of government guarantees when auction is absent. Indeed, when the contract is auctioned, the numerical analysis can follow similar procedures. To facilitate our analysis, suppose that \( d(p, q) = (40 - p + q)e \), where \( e \) is a uniformly distributed random variable with the support on \([0, 1]\). Moreover, based on our assumptions, let us assume that \( c_0(q) = 10 - q, \overline{c}(q, y) = \frac{3}{2}q^2 + \frac{1}{2}y^2, C = 0, \Pi = 29 \), and \( \alpha = 0.6 \).

6.1. Minimum traffic guarantee

In this subsection, the “buyback” price \( b \) is assumed to be \( b = 10 \). By fixing \( q = 5 \) and \( y = 15 \), Fig. 1(a) shows the impact of minimum traffic guarantee on toll charge with guarantee levels of \( v_1 = 7 \) and \( v_2 = 12 \). Note that \( p^*_1(v_1) \) (\( i = 1, 2 \)) makes the private investor obtain the highest profit and both \( \Pi^1_{p}(p^*_1(v_1), q, y) \) and \( \Pi^1_{p}(p^*_1(v_2), q, y) \) are bigger than 29 (the private investor’s reservation utility). Thus, the optimal toll charge under \( v_i \) will be \( p^*_i(v_i) \) (\( i = 1, 2 \)). Moreover, according to Fig. 1(a), both \( p^*_1(v_1) \) and \( p^*_1(v_2) \) are bigger than \( p^*_0 \). Therefore, minimum traffic guarantee improves the optimal toll charge. Fig. 1(a) also shows that \( p^*_1(v_1) < p^*_1(v_2) \), which supports Proposition 1 that the optimal toll charge is increasing in the guarantee level.

By fixing \( p = 25 \) and \( y = 15 \), Fig. 1(b) shows the impact of minimum traffic guarantee on road quality with guarantee levels of \( v_1 = 7 \) and \( v_2 = 12 \). Similarly, Fig. 1(b) shows that the optimal road quality is \( q^*_1(v_1) \) (\( i = 1, 2 \)) and \( q^*_1(v_1) < q^*_0 \) (\( i = 1, 2 \)). Therefore, minimum traffic guarantee reduces the optimal road quality. Moreover, note that \( q^*_1(v_1) > q^*_1(v_2) \), which supports Proposition 1 that the optimal road quality decreases with guarantee level.

By fixing \( p = 25 \) and \( q = 5 \), Fig. 1(c) shows the impact of minimum traffic guarantee on road capacity with guarantee levels of \( v_1 = 7 \) and \( v_2 = 12 \). When \( v_1 = 7 \), the optimal road capacity is \( y^*_1(v_1) \) that is bigger than \( v_1 \). However, when \( v_2 = 12 \), according to Fig. 1(c), we have \( y^*_1(v_2) < v_2 \). Thus, the optimal road capacity will be \( y^*_1(v_2) = v_2 = 12 \), which is consistent with Proposition 2.

6.2. Minimum revenue guarantee

If road quality and road capacity are given by \( q = 5 \) and \( y = 15 \), then based on the expression of \( d(p, q) \) in our numerical example, we have \( F(x, p, q) = \frac{1}{x^2}p \). Thus, to ensure \( F(x, p, q) \leq 1 \) for \( 0 \leq x \leq 15 \), we can obtain that the largest toll price cannot
exceed \( \bar{p} = 30 \). Fig. 2(a) shows the impact of minimum revenue guarantee on toll charge with guarantee levels of \( R_1 = 135 \) and \( R_2 = 146 \). Note that both \( P R_1(p, q, y) \) and \( P R_2(p, q, y) \) are increasing in \( p \) when \( xp \min(y, d(p, q)) \leq R_i \) \( (i = 1, 2) \). Thus the private investor obtains her maximum profit at \( \bar{p} = 30 \). However, Fig. 2(a) shows that \( P R_1(p, q, y) < \bar{p} = 29 \). Therefore, \( R_1 \) fails to attract the private investor. In contrast, when the guarantee level is raised to \( R_2 \), we have \( P R_2(p, q, y) > \bar{p} = 29 \). Consequently, the optimal toll charge is \( p(\geq \bar{p}) \) when \( R = R_2 \).

If toll charge and road capacity are given by \( p = 25 \) and \( y = 15 \), then Fig. 2(b) shows the impact of minimum revenue guarantee on road quality with guarantee levels of \( R_1 = 120 \) and \( R_2 = 150 \). According to Fig. 2(b), both \( P R_1(p, q, y) \) and \( P R_2(p, q, y) \) are decreasing in \( q \) when \( xp \min(y, d(p, q)) \leq R_i \) \( (i = 1, 2) \). Thus, the private investor's profit achieves maximum
at $q = 0$. However, we have $\Pi^K_2(p, 0, y) < \Pi = 29$. This means that minimum revenue guarantee cannot attract the private investor when $R_1 = 120$. In contrast, when $R_2 = 150$, we have $\Pi^K_2(p, 0, y) > \Pi = 29$. Consequently, the optimal road quality will be zero if $R_2 = 150$.

If toll charge and road quality are given by $p = 25$ and $q = 5$, then Fig. 2(c) shows the impact of minimum revenue guarantee on road capacity with guarantee levels of $R_1 = 80$ and $R_2 = 130$. According to Fig. 2(c), the private investor’s expected profit $\Pi^K_2(p, q, y)$ is decreasing in $y$ when $xp\min(y, d(p, q)) \leq R_i$ ($i = 1, 2$) and $\Pi^K_2(p, q, 0) > \Pi = 29$ ($i = 1, 2$). Thus, the optimal road capacity is zero under both $R_1 = 80$ and $R_2 = 130$. 
6.3. Price compensation guarantee

If road quality and road capacity are given by \( q = 5 \) and \( y = 15 \), then Fig. 3(a) shows the impact of price compensation guarantee on toll charge with guarantee levels of \( u_1 = 5 \) and \( u_2 = 15 \). According to Fig. 3(a), the private investor’s profit under \( u_1 = 5 \) and \( u_2 = 15 \) achieves maximum at \( p_1(u_1) \) and \( p_2(u_2) \), respectively. Moreover, we have \( \Pi^i(p_i(u_i), q, y) > \Pi = 29 \) (\( i = 1, 2 \)). Therefore, the optimal toll charge under \( u_i \) will be \( p_i(u_i) \) (\( i = 1, 2 \)). Fig. 3(a) shows that both \( p_1(u_1) \) and \( p_2(u_2) \) are smaller than \( p_0 \). This supports our argument that price compensation guarantee reduces toll charge. Moreover, we have \( p_2(u_2) < p_1(u_1) \). Consequently, Proposition 4 is supported that the optimal toll charge under price compensation guarantee decreases with guarantee level.

If toll charge and road capacity are given by \( p = 25 \) and \( y = 15 \), then Fig. 3(b) shows the impact of price compensation guarantee on road quality with guarantee levels of \( u_1 = 5 \) and \( u_2 = 15 \). Similar to the above analysis, the private investor's
optimal road quality will be \( q_i^1(u_i) \) (i = 1, 2). According to Fig. 3(b), we have \( q_i^1(u_i) > q_0^1 \) (i = 1, 2) and \( q_i^2(u_i) > q_0^1 \), which implies that price compensation guarantee improves road quality and the optimal road quality increases with the guarantee level.

If toll charge and road quality are given by \( p = 25 \) and \( q = 5 \), then Fig. 3(c) shows the impact of price compensation guarantee on road capacity with guarantee levels of \( u_1 = 5 \) and \( u_2 = 15 \). In parallel with the analysis of road quality, \( y_i^3(u_i) > y_0^1 \) (i = 1, 2) and \( y_i^3(u_i) > y_0^1 \) also support our argument that price compensation guarantee improves road capacity and the optimal road capacity increases with the guarantee level.

7. Conclusion

Government guarantees play an important role in attracting the private investor’s participation into the BOT road project. However, their impact on the project is largely overlooked by previous literature. We fill this theoretical gap by modeling the impact of three government guarantee types – minimum traffic guarantee, minimum revenue guarantee, and price compensation guarantee – on the private investor’s decisions on toll charge, road quality, and road capacity.

Our model results show that minimum traffic guarantee increases toll charge while decreasing road quality. When the guarantee level is low, minimum traffic guarantee has no impact on road capacity. Otherwise, minimum traffic guarantee can increase road capacity. Under minimum revenue guarantee, if the guarantee level is sufficiently high, the private investor will concentrate on minimizing her construction and O&M costs to obtain more profit. As a result, the optimal toll charge will be sufficiently large and road quality and road capacity will approach zero. Under price compensation guarantee, however, the optimal toll charge is decreased and both road quality and capacity are increased. This paper further investigates the impact of government guarantees when the contract is auctioned. We find that auction reduces the impact of government guarantees on toll charge. However, auction cannot affect the impact of government guarantees on road quality and road capacity.

This paper contributes to practice by shedding light on how the government should design government guarantees in the BOT road project. Currently, when choosing guarantee types and guarantee levels, the government mainly focuses on the financial evaluation with the assumption that they have a minimal or no impact on the project. However, this paper concludes that different government guarantees affect project characteristics differently. Therefore, we suggest that in designing government guarantees, the government should take into account their impact on the project. This paper also implies that the government can employ government guarantees as a tool to encourage the private investor to invest in road quality and road capacity. For instance, based on our model results, if the government intends to elevate road quality, she should provide the private investor with price compensation guarantee or minimum traffic guarantee with a low guarantee level.

Moreover, this paper can provide some policy implications for the employment of auction in the BOT road project. It has long been recognized that auction reduces the toll charge. Our model results also support such a recognition. Furthermore, we suggest another impact of auction. This paper shows that auction can reduce the impact of government guarantees on toll charge while having no impact on road quality and capacity. Thus, if the government needs a smaller change of toll charge when government guarantees are provided, then auction should be preferred.

It is worth pointing out that, in this paper, we assume that the private investor has freedom to choose toll charge, road quality, and road capacity. However, in practice these decisions are, more often than not, regulated by the government: The private investor’s decisions on toll charge, road quality, and road capacity should satisfy the government’s general requirements. Therefore, our model framework does not accurately represent most BOT practices. If we relax our assumption to incorporate the government’s requirements, then mathematically the private investor’s decisions will become a constrained optimization problem. In doing so, our discussion would be substantially complicated. However, our model results will not change much. For instance, under minimum traffic guarantee without auction, let us denote the price cap by \( p_0 \) and the government’s minimum standards of road quality and road capacity by \( q_q \) and \( y_q \), respectively. Recall that the private investor’s profit function is assumed to be strictly concave with respect to \( p, q \), and \( y \). Thus, the private investor’s optimal decisions on toll charge, road quality, and road capacity under government regulation will become \( \min(p_0, p), \max(q_q, q), \) and \( \max(y_q, y) \), respectively. Therefore, if the government’s price cap is low and standards of road quality and capacity are high, then the private investor’s optimal decisions will become the government’s pre-determined standards, which makes our problem trivial. Otherwise, our model results in Proposition 1 and 2 still hold that characterize the impact of minimum traffic guarantee on toll charge, road quality, and road capacity.

There are some limitations in this paper. First, the one-dimensional award criterion is employed here. However, in some modern toll roads, various criteria are considered such as road quality, road capacity, delivery date or financial strength. Also, we assume that there are no other competitive roads around the tendered BOT road project. However, in many cases, the tendered BOT road project is part of the existing road network, which inevitably influences the private investor’s toll charge and her decisions on road quality and capacity. Further analysis can be conducted by embedding the project into the existing road network. Finally, we assume that demand only depends on price and quality. However, many other factors may also play a role in determining traffic demand such as travel time. We believe further analysis by taking these limitations into account can derive more general results.
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Appendix A

Proof of Proposition 1. Suppose that \( v_1 \leq v_2 \). Accordingly, the optimal toll charge and road quality are \( p^*_1(v_1), q^*_1(v_1) \) and \( p^*_2(v_2), q^*_2(v_2) \), respectively. Therefore, \( k^*_1(p^*_1(v_1)) = 0, k^*_2(p^*_2(v_2)) = 0, h^*_1(q^*_1(v_1)) = 0, \) and \( h^*_2(q^*_2(v_2)) = 0 \). Based on assumptions that \( F_p(x, p, q) < 0 \) and \( F_q(x, p, q) > 0 \), we have \( \int_0^{v_1} F_p(x, p, q) dx \leq \int_0^{v_1} F_p(x, p, q) dx \) and \( \int_0^{v_2} F_q(x, p, q) dx \leq \int_0^{v_2} F_q(x, p, q) dx \). Therefore, we can obtain that \( k^*_1(p^*_1(v_1)) \leq k^*_2(p^*_2(v_2)) \) for any \( p \) and \( h^*_1(q^*_1(v_1)) \geq h^*_2(q^*_2(v_2)) \) for any \( q \). Specifically, \( k^*_1(p^*_1(v_1)) \leq k^*_2(p^*_2(v_1)) \) and \( h^*_1(q^*_1(v_1)) \geq h^*_2(q^*_2(v_2)) \). Note that \( k^*_1(p^*_1(v_1)) = k^*_2(p^*_2(v_2)) = 0 \) and \( h^*_1(q^*_1(v_1)) = h^*_2(q^*_2(v_2)) = 0 \). Thus, \( k^*_1(p^*_1(v_1)) \leq k^*_2(p^*_2(v_1)) \) and \( h^*_1(q^*_1(v_1)) \geq h^*_2(q^*_2(v_1)) \). Based on the assumption that the private investor’s expected profit is strictly concave with respect to \( p \) and \( q \), we can get that \( p^*_1(v_1) \leq p^*_1(v_2) \) and \( q^*_1(v_1) \geq q^*_1(v_2) \).

By the expression of \( g^*_1(y) \), we can easily get that \( y^*_1(v) \) is independent of \( v \). □

Proof of Proposition 2. Note that \( p^*_0 = p^*_1(0), q^*_0 = q^*_1(0), \) and \( y^*_0 = y^*_1(0) \). The conclusion immediately follows by Proposition 1. □

Proof of Proposition 3. When the guarantee level is sufficiently high, the private investor’s expected profit is \( \Pi^*_2(p, q, y) \). Taking derivative of \( \Pi^*_2(p, q, y) \) with respect to \( p \), we can get

\[
k^*_2(p) = \frac{\partial \Pi^*_2(p, q, y)}{\partial p} = -\alpha c_0(q) \int_0^y F_p(x, p, q) dx > 0.
\]

Thus, when the guarantee level is sufficiently high, the optimal toll charge in this case will be sufficiently large.

Taking derivative of \( \Pi^*_2(p, q, y) \) with respect to \( q \) yields

\[
h_2(q) = \frac{\partial \Pi^*_2(p, q, y)}{\partial q} = -\alpha c_0(q) \int_0^y F_p(x, p, q) dx - \alpha c_0(q) \int_0^y F_q(x, p, q) dx - \delta^a \int_{y_{m-1}}^y \omega(q, x) dx.
\]

Recall the assumption that \( \frac{\partial c_0(q) \min y \partial d(p, q)}{\partial q} \) < 0. Thus, \( h_2(q) < 0 \). Consequently, \( \Pi^*_2(p, q, y) \) achieves maximum at \( q = 0 \). We next show that \( xp \min(y, d(p, 0)) \leq R \). Note that \( xp \min(y, d(p, q)) \) is increasing in \( q \). Therefore,

\[
xp \min(y, d(p, 0)) \leq xp \min(y, d(p, q)) \leq \max \{xp \min(y, d(p, q))\} \leq R.
\]

Hence, we can conclude that the optimal road quality will be zero.

Taking derivative of \( \Pi^*_2(p, q, y) \) with respect to \( y \) gives

\[
g_2(y) = \frac{\partial \Pi^*_2(p, q, y)}{\partial y} = -\alpha c_0(q) F(y, p, q) - \delta^a \int_{y_{m-1}}^q \omega(s, y) ds < 0.
\]

Note that \( xp \min(y, d(p, q)) \) is increasing in \( y \). Thus, similar to the above analysis, the optimal road capacity will also approach zero if \( \max \{xp \min(y, d(p, q))\} \leq R \). □

Proof of Proposition 4. Suppose that \( u_1 \leq u_2 \). Then, the optimal toll charge, road quality, and road capacity are \( p^*_1(u_i), q^*_1(u_i), y^*_1(u_i) \) \( i = 1, 2 \), respectively. Moreover, we have \( k^*_1(p^*_1(u_i)) = 0, h^*_1(q^*_1(u_i)) = 0, \) and \( g^*_1(y^*_1(u_i)) = 0 \) \( i = 1, 2 \). Based on the assumption that \( F_p(x, p, q) < 0, F_q(x, p, q) > 0 \) and \( F(x, p, q) \geq 0 \), we can get that \( k^*_1(p^*_1(u_i)) \geq k^*_1(p^*_2(u_2)) \), \( h^*_1(q^*_1(u_i)) \leq h^*_1(q^*_2(u_2)) \), and \( g^*_1(y^*_1(u_i)) \leq g^*_1(y^*_2(u_2)) \). Note that \( k^*_1(p^*_1(u_1)) = k^*_1(p^*_2(u_2)) = 0, h^*_1(q^*_1(u_1)) = h^*_1(q^*_2(u_2)) = 0, \) and \( g^*_1(y^*_1(u_1)) = g^*_1(y^*_2(u_2)) = 0 \). Thus, \( k^*_1(p^*_1(u_1)) \geq k^*_1(p^*_2(u_2)) \), \( h^*_1(q^*_1(u_1)) \leq h^*_1(q^*_2(u_2)) \), and \( g^*_1(y^*_1(u_1)) \leq g^*_1(y^*_2(u_2)) \). Recall the assumption that \( \Pi_2(p, q, y) \) is strictly concave with respect to \( p, q, \) and \( y \). We can obtain that \( p^*_1(u_1) \geq p^*_1(u_2), q^*_1(u_1) \leq q^*_1(u_2), \) and \( y^*_1(u_1) \leq y^*_1(u_2) \). □

Proof of Proposition 5. Note that \( p^*_0 = p^*_1(0), q^*_0 = q^*_1(0), \) and \( y^*_0 = y^*_1(0) \). The conclusion immediately follows by Proposition 4. □
Proof of Proposition 7. We have $F_p(v, p, q) \int_p^\infty t(v) dr - F(v, p, q)t(p) > 0$ based on the assumption that $F(v, p, q) \int_p^\infty t(v) dr$ strictly increases with $p$. Note that $\frac{\partial}{\partial p} [F_p(v, p, q) \int_p^\infty t(v) dr - F(v, p, q)t(p)]$ is the first derivative of $\tilde{k}_p^v(p)$ with respect to $v$.

Thus, $\tilde{k}_p^v(p)$ also strictly increases with $v$. Suppose that $v_1 < v_2$. Then we can get $\tilde{k}_p^v(p_1(v_1)) < \tilde{k}_p^v(p_2(v_1))$. Note that $\tilde{k}_p^v(p_1(v_2)) = \tilde{k}_p^v(p_2(v_1)) = 0$. Thus, $\tilde{k}_p^v(p_2(v_2)) < \tilde{k}_p^v(p_2(v_1))$. Based on the assumption of strictly concave profit function, we have $p_1(v_2) > p_1(v_1)$.

For any $v$, we have $\tilde{k}_p^v(p_1(v)) = 0$ and $\tilde{k}_p^v(p_1(v)) = 0$. Thus, based on the expression of $\tilde{k}_p^v(p)$, we get

$$k_p^v(p_1(v)) = \frac{\Pi_p^v(p_1(v), q, y)\tilde{p}(p_1(v))}{1 - \tilde{p}(p_1(v))} = \Pi_p^v(p_1(v), q, y)\tilde{p}(p_1(v)).$$

Recall the assumption that the private investor’s expected profit $\Pi_p^v(p, q, y)$ is strictly concave with respect to $p$. Thus, $k_p^v(p)$ is strictly decreasing in $p$. As a result, we can write that $p_1(v) = (k_p^v)^{-1}(0)$ and $p_2(v) = (k_p^v)^{-1}(\Pi_p^v(p_1(v), q, y)\tilde{p}(p_1(v)))$, where $(k_p^v)^{-1}(\cdot)$ represents the inverse function of $k_p^v(p)$. Hence, $p_1(v) - p_2(v) = (k_p^v)^{-1}(0) - (k_p^v)^{-1}(\Pi_p^v(p_1(v), q, y)\tilde{p}(p_1(v)))$.

Taking derivative of $p_1(v) - p_2(v)$ with respect to $v$, we have

$$\frac{d(p_1(v) - p_2(v))}{dv} = \frac{d(k_p^v)^{-1}(0)}{dv} < 0.$$ 

Because $p_1(v) > p_2(v)$, we have $\frac{d(k_p^v)^{-1}(0)}{dv} = k_p^v(p_1(v)) > k_p^v(p_2(v)) = 0$. Recall our assumption that the hazard rate $\tilde{p}(v)$ is strictly increasing in $p$, as well as $\frac{d(k_p^v)^{-1}(0)}{dv} < 0$ and $\frac{d\tilde{p}(p_1(v) - p_2(v))}{dv} > 0$. We can obtain that $\frac{d(p_1(v) - p_2(v))}{dv} > 0$. □

Proof of Proposition 8. We have

$$\alpha p^2(R) \min(y, d(p, d(p, R, q))) \leq \max(\zeta p \min(y, d(p, q))) \leq R$$

for any given $q$ and $y$. Therefore, based on the assumption that the private investor’s expected profit $\Pi_p^v(p, q, y)$ is strictly concave with respect to $p$, the optimal toll charge in this case will be $p_2(R)$.

Suppose that $R_1 \leq R_2$, then we have $k_p^v(p_2(R_1)) = k_p^v(p_2(R_2)) = 0$. Note that

$$k_p^v(p) = k_p^v(p)\tilde{p}(p) - \Pi_p^v(p, q, y)t(p),$$

and $\Pi_p^v(p, q, y) \leq \Pi_p^v(p_1(v), q, y) \leq \Pi_p^v(p_1(v), q, y)$. Consequently, we can derive that $p_2(R_2) \leq p_2(R_1)$. □

Proof of Proposition 9. If $u_1 \leq u_2$, then $k_p^v(p_2(u_1)) = k_p^v(p_2(u_2))$. Therefore, we have $k_p^v(p_2(u_1)) = k_p^v(p_2(u_2)) = 0$. Thus, we have $k_p^v(p_2(u_2)) \geq k_p^v(p_2(u_1))$. Hence, based on the assumption of concave profit function, $p_2(u_1) \geq p_2(u_2)$. Following the proof of Proposition 7, the conclusion follows that $p_2(u) - p_2(u)$ decreases with $u$. □

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